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Introduction

The National Council of Teachers of Mathematics (NCTM) Standards advocate a unified approach to mathematics education incorporating multiple strands in coherent focused elements. Emphasis is placed on the use of technology, visual thinking, and the connection between geometry and algebra. The notion of integrating algebra and geometry is at the forefront of mathematics education in the US today, potentially transformative of mathematics education in the 21st century when technology integration is not only reasonable because of access but also because of the nature of the impact of mathematics on knowledge and directions in this century. While geometry and algebra systems have existed separately for the last couple of decades, an application that unifies the two is available through a brand new technology. This unification facilitates a variety of well-understood goals:

- The user interface must be simple so that students can focus on the reasoning and problem solving rather than struggle with the technology.

- The application must enable exploration.

- As much as possible, the technology must blur the artificial separation between the algebra and the geometry.

- It must provide easy access but also be challenging. That is, it must be easily used by students at a wide range of performance levels while also being capable of providing challenge to the most able students.

- It must address traditional concepts in addition to facilitating the use of more modern and realistic approaches to geometry and to real applications.

- The mathematical concepts addressed must be expanded in order to make a major impact on the high school curriculum.

The technology that meets these goals is now available. *Geometry Expressions* developed by Saltire Software, is a computer application that, unlike other interactive geometry systems, can automatically generate algebraic expressions from geometric figures. A simple algebra system is embedded directly into *Geometry Expressions*. Generation and simplification of expressions along with simple algebraic manipulations are handled inside *Geometry Expressions*. From a pedagogical standpoint an interactive symbolic geometry system affords a remarkable opportunity to make concrete the concept of a variable in a readily identifiable real world setting. An educational tool takes advantage of this opportunity if it provides outputs in a symbolic form. These outputs are algebraic expressions using the input parameters.

Our goal was to use *Geometry Expressions* to develop a set of problems for second year algebra and/or precalculus courses addressing the topics of geometric transformations of functions and optimization. In this volume you will find eight interactive problems with different levels of difficulty. In each problem the main focus is on the development of the students' ability to recognize and make connections using multiple representations of the same object, such as geometric shape and function. By connecting geometric and algebraic representations the student develops a more thorough understanding of the problem.

Discovering Parabolas

Teaching Goals:

1. Students verify that a unique parabola with the equation $y = ax^2 + bx + c$, $a \neq 0$, exists through any three given points if and only if these points are not collinear.

2. Students verify that an infinite number of parabolas with the equation $y = ax^2 + bx + c$, $a \neq 0$, exist through any two given points.

3. Students should be able to construct the parabola defined by the coordinates of three points and find its equation with the help of the software.

4. Students should be able to construct this parabola by geometric transformations from the graph of the parabola $y = x^2$ and confirm the equation given by the software.

Prior Knowledge

*   Students should know the graph of quadratic functions in two forms: $y = ax^2 + bx + c$ and $y = a(x - x_0)^2 + y_0$, $a \neq 0$, and they should know this graph has the shape of a parabola.

*   Students should know geometric representations and coordinate forms of the following transformations: translation along the coordinate axes, reflection about coordinate axes, and dilation along coordinate axes.
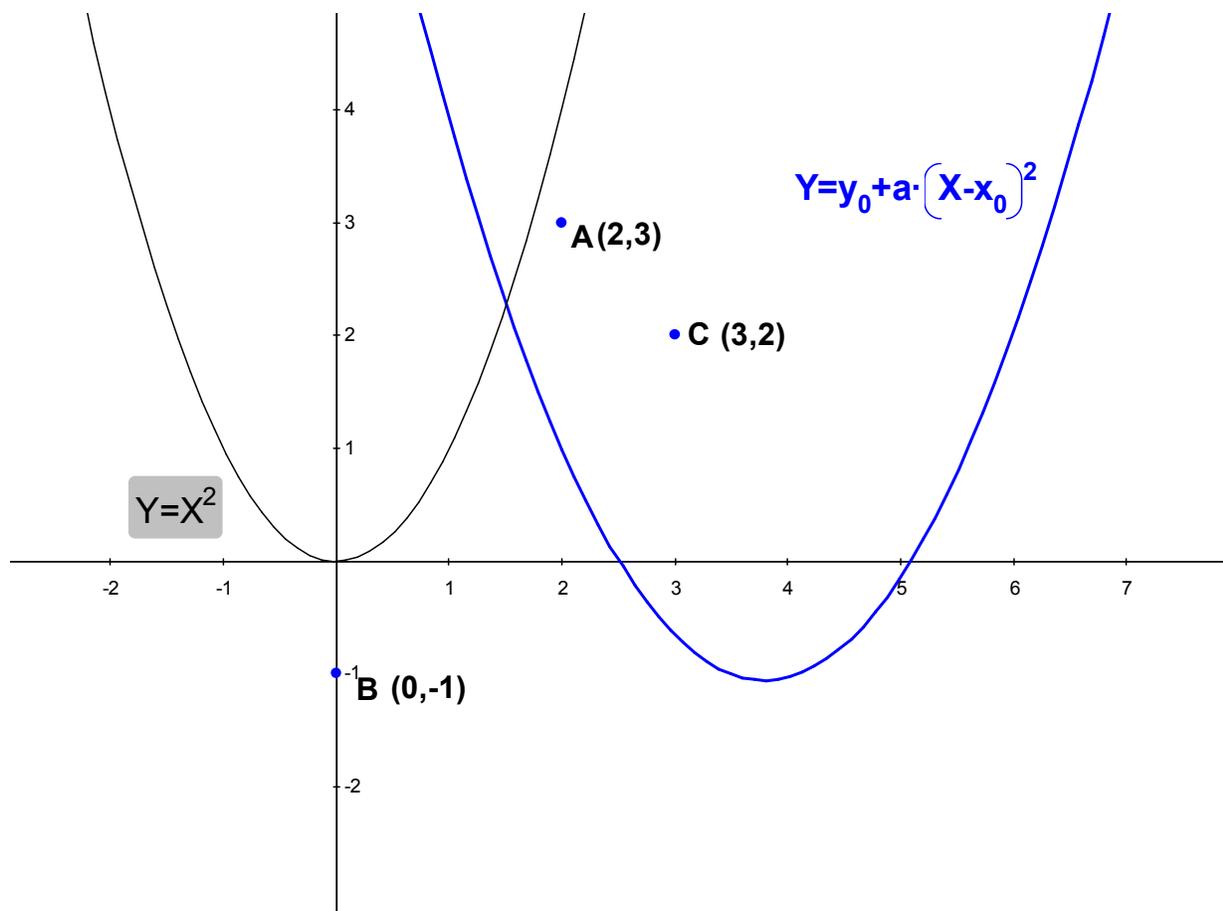
Problem:

This is a problem of the existence of a quadratic function (a parabola) as a graph. Since we consider graphs of functions, situations when two points are located on the same vertical line are not considered.

Part 1 – Parabola by 3 Points

In this part of the problem students are given three specific points, the maximum, with coordinates (2, 3), and other points with coordinates (0, -1) and (3, 2). They are also given the graph of an initial parabola $y = x^2$ and the graph of a general parabola $y = a(x - x_0)^2 + y_0$.

1. Open a new file. If axes do not appear in the blank document, create them by clicking the **Toggle grid and axes** icon on the top toolbar.

2. Select **Function** from the **Draw** toolbox. Type *x^2* in the **Function Type** dialog.

3. Select **Function** from the **Draw** toolbox. Type *y[0]+a*(X-x[0])^2* for the function. Click the graph of the function to highlight it, right click the mouse to invoke the context menu, and select **Properties**. From the **Display Properties** dialog choose <u>Line Color</u> – Blue.

4. Select **Point** from the **Draw** toolbox. Construct three points anywhere on the screen. Select point A, choose **Coordinate** from the **Constrain** toolbox and type coordinates (2, 3) for the point A, press *Enter*. Repeat this operation to constrain points B and C. If desired, change the font, size and color for the labels of the points.

$$Y=y_0+a\cdot\left(X-x_0\right)^2$$

A(2,3)

• C (3,2)

$$Y=X^2$$

• B (0,-1)

Q1. Can you find a parabola that has an equation with integer numerical coefficients and will go through these three points? If it exists, what is the equation of this parabola? In order to do that:
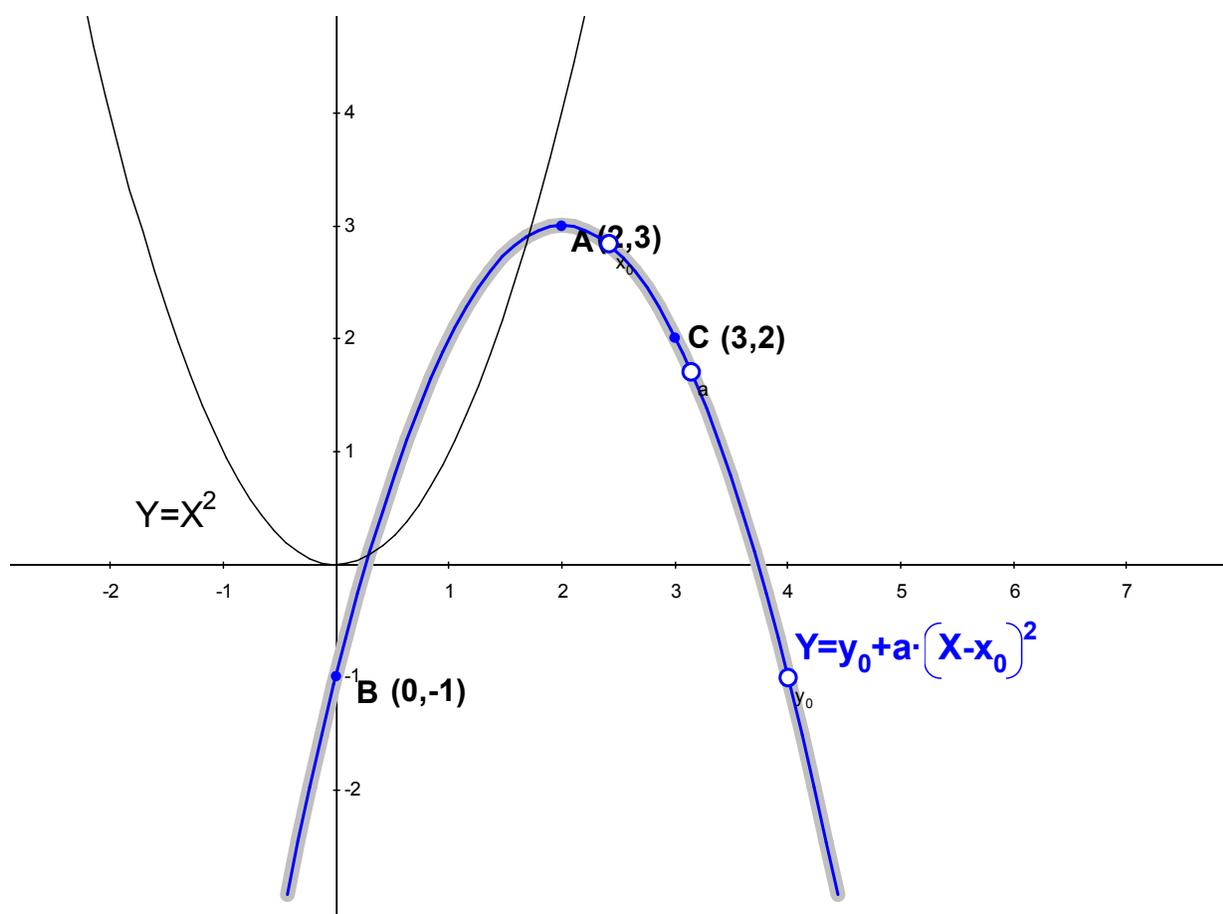
Q2. Investigate which parameters in the equation of the parabola $y = y_0 + a(x - x_0)^2$ are responsible for which type of transformation of the graph.

1. Click-and-drag the parabola. When you move the parabola, a small circle appears on the curve. We refer to this as the "drag handle". A variable name labels the drag handle. Note which way you are able to move the curve with the parameterized handle (click-and-drag the handle).

2. Click another part of the parabola and another drag handle appears on the curve with another variable label. Create three drag handles for the three variables of the equation. Observe how each handle / variable is responsible for a type of transformation of the curve.

3. Record your observations in the table below:

Parameter in the equation $y = y_0 + a(x - x_0)^2$	Transformation (vertical translation, horizontal translation, dilation)
a	
Y_0	
X_0	

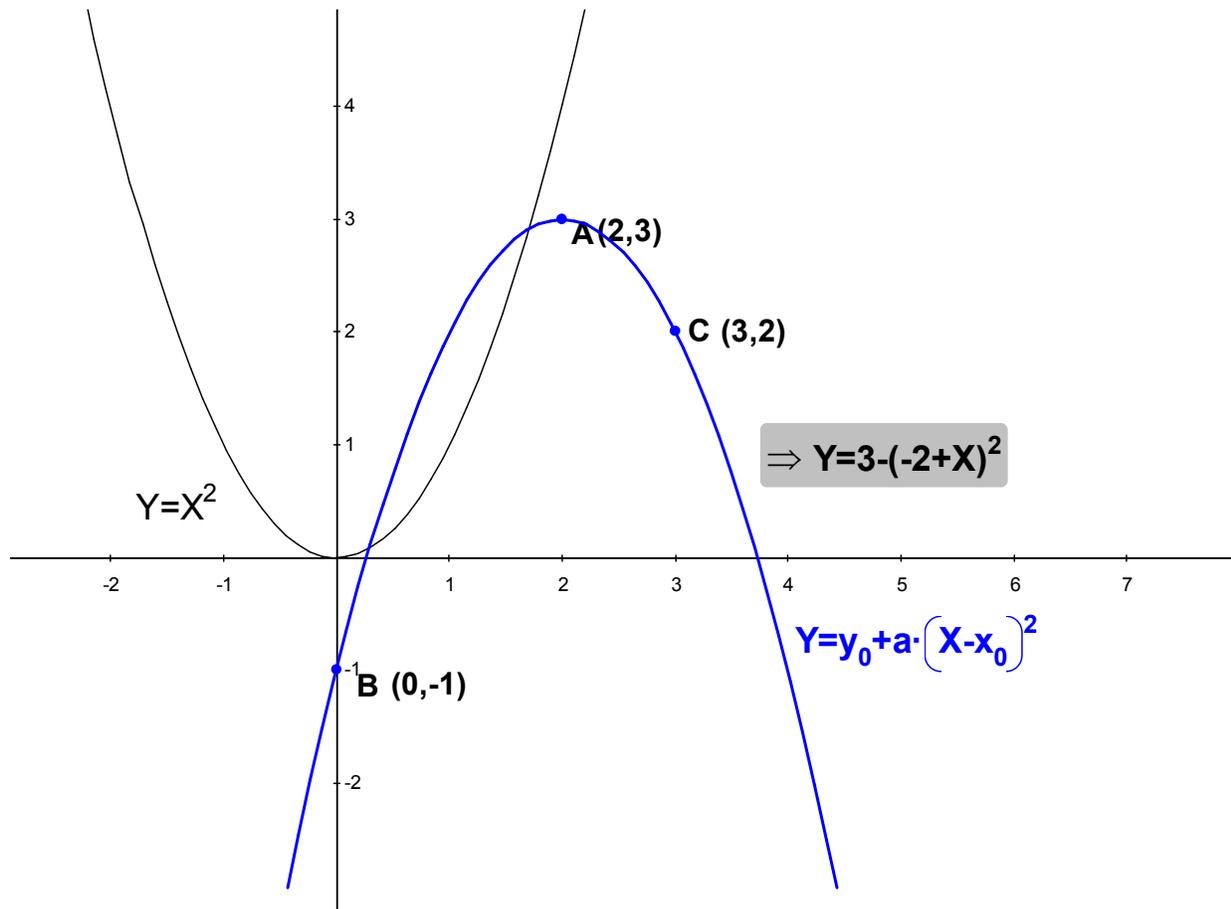**A.** a – dilation, y_0 – vertical translation, x_0 – horizontal translation

Q3. Drag and move parabola until it passes through the points A, B, and C. Construct your conjecture about the equation of the parabola (round numerical values to integers).

Students should vary parameters a, x_0, and y_0 to try to complete the task. Observe the values of the parameters in the **Variables** toolbox. When they think they have the solution they can round the numeric values of the coefficients with the help of the software and verify their equation by substitution. In the figure above, $a \approx -0.998, x_0 \approx 2.003, y_0 \approx 3.009$.

A. The equation of the parabola is: $y = 3 - (x - 2)^2$. Here are steps for verification:

1. In the **Variables** toolbox, enter the predicted values for the parameters, $a = -1, x_0 = 2, y_0 = 3$. Change a variable value from the toolbox by clicking the variable row and entering a new value in the data entry window below the list.

2. Click the graph of the parabola, then from the **Calculate** toolbox click the **Real** tab and the **Implicit equation** icon. The numeric equation for the parabola will appear on the screen
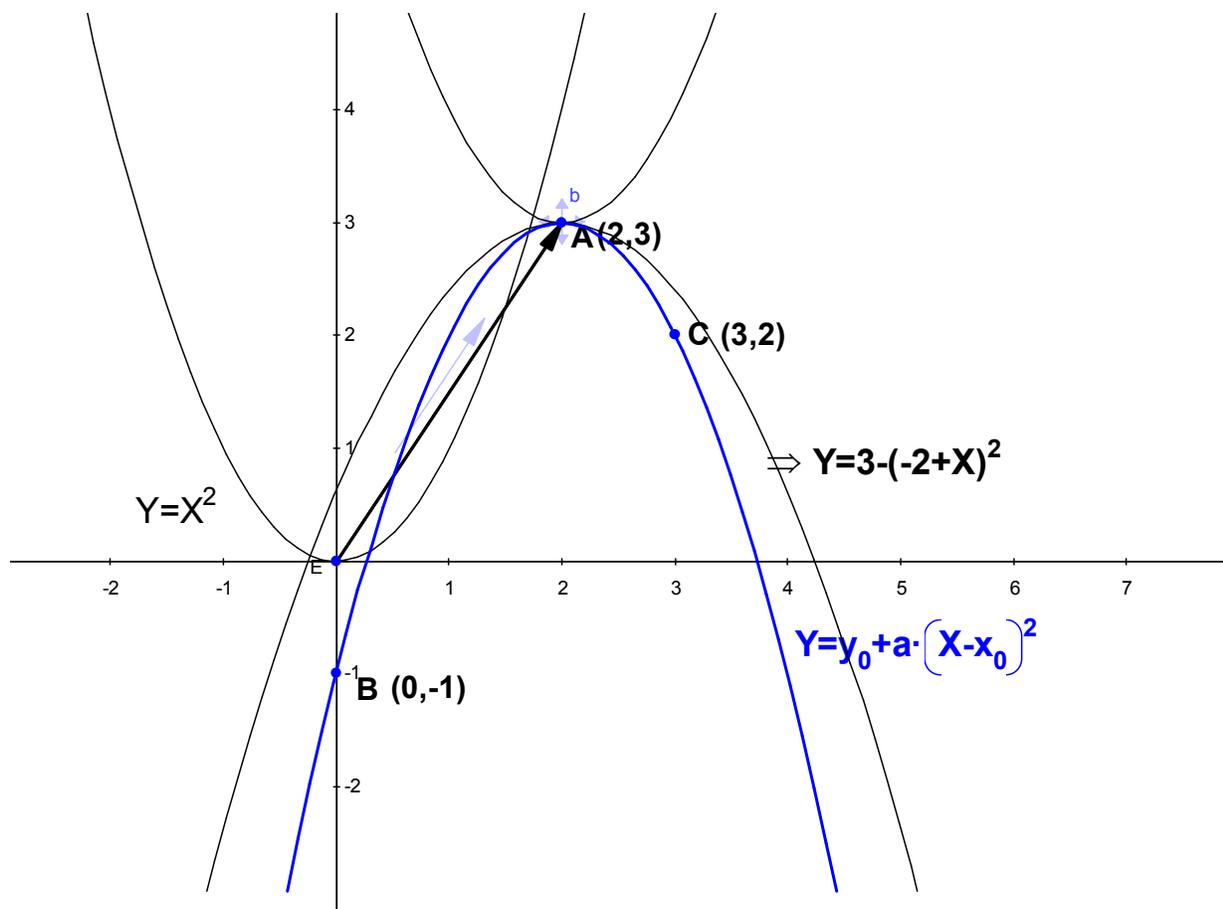

3. Substitute values of all three points and verify that all three points lie on this curve.

Q4. Find geometric transformations that transform the graph of $y = x^2$ into the graph of the parabola in the last problem. Is it important to preserve the order of transformations? Verify your answers with the help of the software.

A. The transformations are: translation by the vector (2, 3) and dilation by the factor -1. These are commutating. Students should complete the construction, verify coincidence of the image with the parabola and the fact that the order of transformation does not affect the result. Here are the steps of the construction:

1. Select **Vector** from the **Draw** toolbox and construct a vector from the origin to point *A*.

2. Click the graph of the parabola $y = x^2$, select **Translation** from the **Construct** toolbox and then click the vector. The image of the translated parabola will appear on the graph.

3. Click the translated curve, select **Dilation** from the **Construct** toolbox. Select point *A* and leave *b* in the data entry box for the dilation factor (press *Enter* to accept the value).

4. In the **Variables** toolbox setup a range of values for the dilation factor *b* from -2 to 0. Click variable *b* (the row will be highlighted), enter *-2* in the <u>Minimum value</u> box in the lower left corner and *0* in the <u>Maximum value</u> box in the lower right corner. Drag the slider bar to adjust the curve so that it coincides with the graph of the parabola $y = 3 - (x - 2)^2$. The value of *b* should be -1.

Q5. Do you think there exists another parabola passing through points A, B and C? Explain your answer.

A. Three non-collinear points define a parabola uniquely. This can be proved analytically.

Part 2 – The Existence of a Parabola Passing Through Three Arbitrary Points
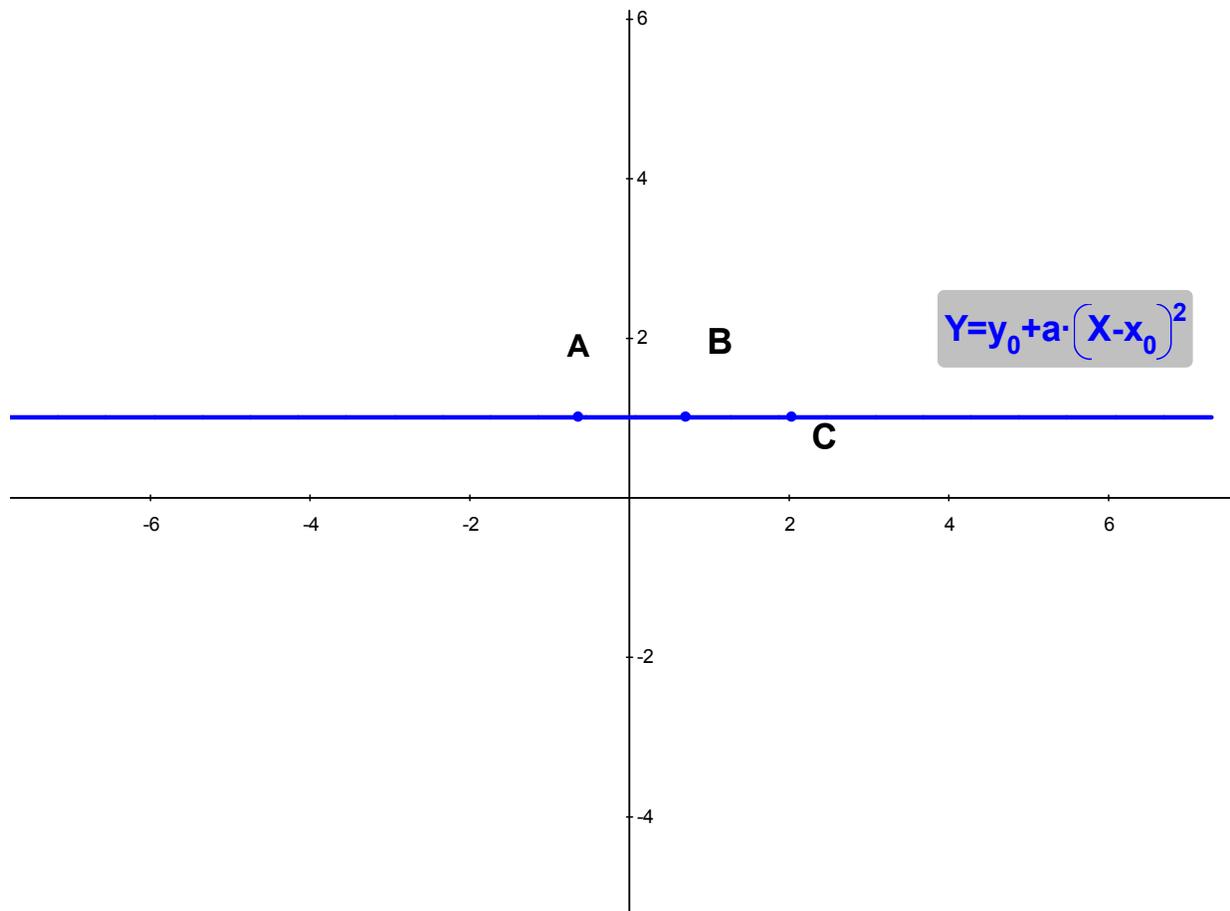
In this part of the problem students investigate how the relative positions of three arbitrary points affects the existence of a parabola passing through these points. Given three points on the plane, A, B, and C , and a parabola defined by the equation $y = y_0 + a(x - x_0)^2$.

1. Open a new file. If the axes do not appear in the blank document, create them by clicking the **Toggle grid and axes** icon on the top toolbar.

2. Select **Function** from the **Draw** toolbox. Type *y[0]+a*(X-x[0])^2* for the function. Click the graph of the function to highlight it, right click the mouse to invoke the context menu, and select **Properties**. From the **Display Properties** dialog choose Line Color – Blue, Line Style – Solid 2.

3. Select **Point** from the **Draw** toolbox. Construct three points anywhere on the screen.

4. Select point *A* and the parabola. Select **Incident** from the **Constrain** toolbox and point *A* will appear on the curve. Repeat the same procedure with the points B and C.
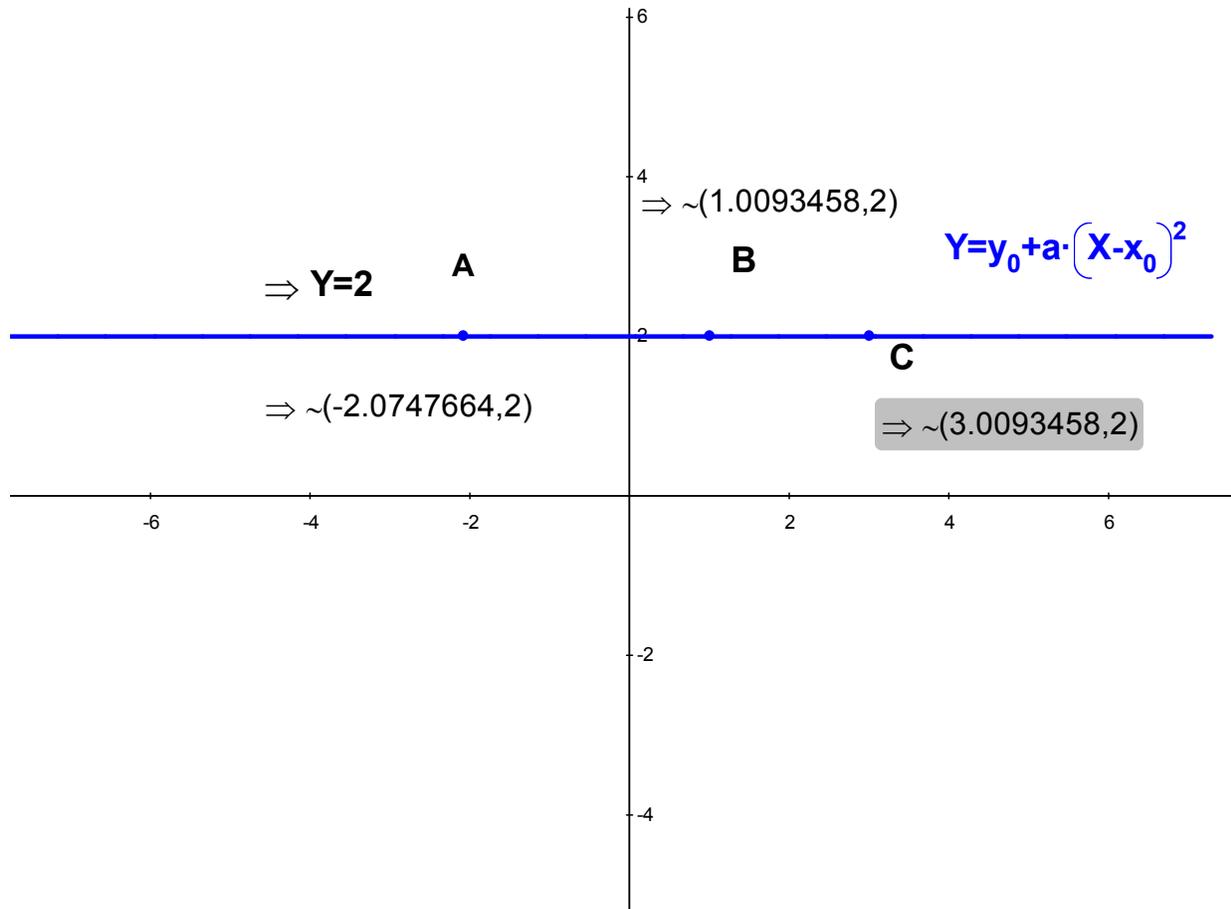
Q1. How can you position points A, B, and C, so that the parabola no longer exists, *e.g.* the shape of the parabola changes into a different shape?

A. When points lie on the same line, the parabola becomes a line. Since we use the equation of parabola in the form $y = y_0 + a(x - x_0)^2$ the line will always be horizontal in the form *y = constant* and *a = 0*. Students can verify this with following the steps:

1. Move all points to lie on the line *y* = some constant (approximately). The parabola will look like a horizontal line.

2. Observe the value of parameter *a* in the **Variables** toolbox. The value of *a* is close to zero. Assume that *a = 0* and enter 0 for the value of *a*.

A   B   C

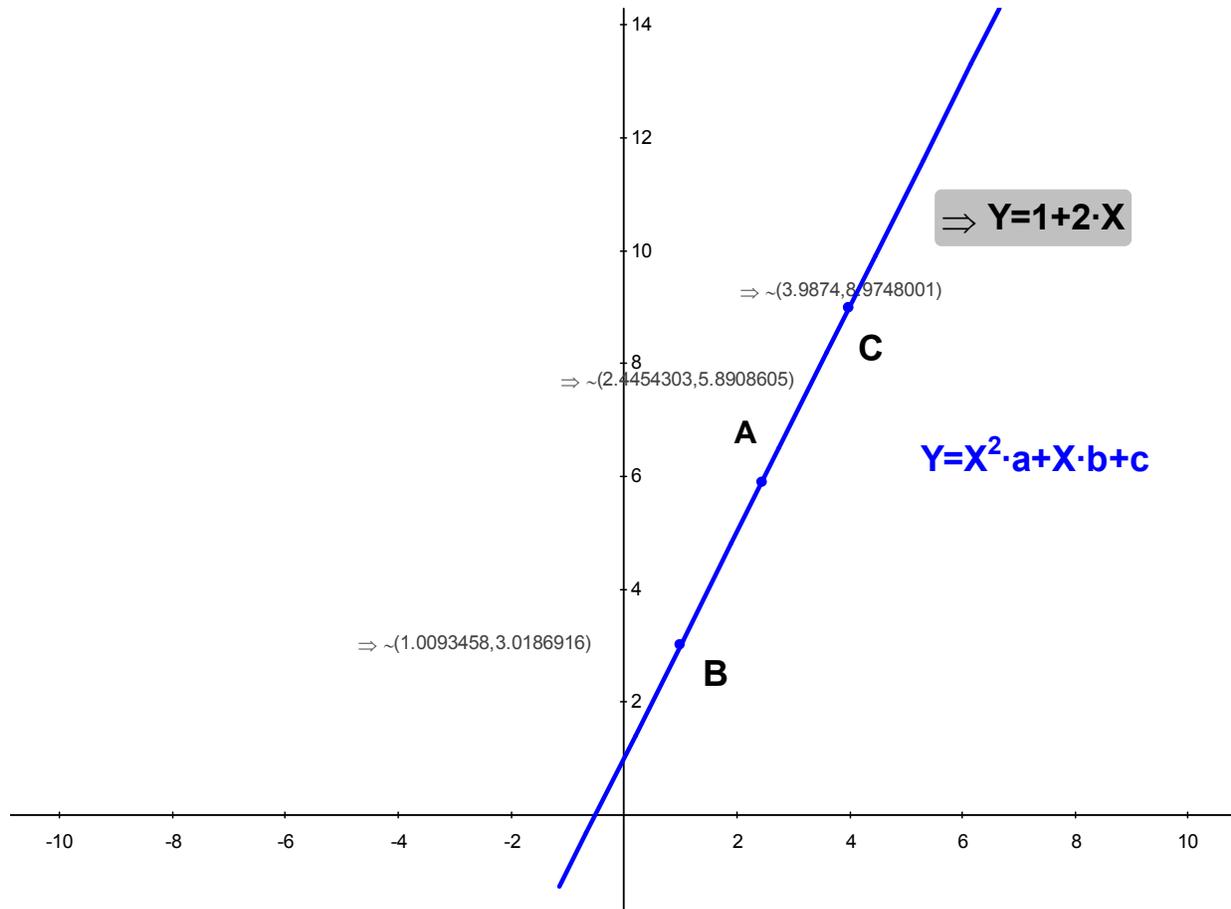$$Y = y_0 + a \cdot \left(X - x_0\right)^2$$

3. Check the real coordinates of points A, B, and C and. Make sure to use the **Real** tab in the **Calculate** toolbox. Select each point, one at a time and choose **Coordinates** from the **Calculate** toolbox.

4. Click the line, and select **Implicit equation** (**Real** tab). The equation of the line will appear on the screen confirming the fact that the line is not a quadratic function.

$\Rightarrow$ ~(1.0093458,2)

B

$Y=y_0+a\cdot\left(X-x_0\right)^2$

$\Rightarrow$ **Y=2**   **A**

$\Rightarrow$ ~(-2.0747664,2)

C

$\Rightarrow$ ~(3.0093458,2)

Extensions:

1. The existence of a parabola: given a parabola defined by the equation $y = ax^2 + bx + c$ and three points, A, B, C, follow steps similar to those above to determine that if three points are collinear, then *a* = 0, when the points lie on an inclined line with slope *b*, and therefore the parabola does not exist.

⇒ Y=1+2·X

⇒ ~(3.9874,8.9748001)

C

⇒ ~(2.4454303,5.8908605)

A

Y=X²·a+X·b+c

⇒ ~(1.0093458,3.0186916)

B

2. The existence of a parabola through any two points: given two points, the choice of a 3rd point defines a unique parabola, so there are an infinite number of parabolas since there are an infinite number of choices for the 3rd point. In this case the teacher can provide students with conditions based on real problems, for example:

a) given the point of a basketball shot and the position of the basket, make the basket

b) make a pass with a volleyball at its highest point (vertex)

c) trace a soccer ball kick so that it hits the right upper corner of the goal.